Second order homogeneous linear ODE.

A second order linear ODE is

$$y'' + p(x)y' + q(x)y = r(x) \rightarrow (1)$$

(i) if $r(x) = 0$, then equ^n (1) is called homo. linear ODE.

(ii) if $r(x) \neq 0$, " " non-homo linear ODE.

Ex: (i) $y'' + 2y' + y = 0$ (homo.)
(ii) $y'' + y = \tan x$ (non homo.)

Superposition principle.

A second order homogeneous linear ODE is

$$y'' + p(x)y' + q(x)y = 0 \rightarrow (1)$$

Let, y_1 and y_2 be sol^{n}s of (1). Then linear comb^n of y_1 and y_2 i.e. $c_1y_1 + c_2y_2$ ($c_1, c_2 \rightarrow$ arb. const) is also a sol^n of equ^n (1).

Ex. Let, $\cos x$ and $\sin x$ are sol^{n}s of homo. linear ODE $y'' + y = 0$

~~Then~~,

Sol^n: Verification: Let, $y_1 = \cos x$, $y_2 = \sin x$

$y_1'' = -\cos x$; $y_2'' = -\sin x$

Now, $y_1'' + y_1 = -\cos x + \cos x = 0$.

Again, $y_2'' + y_2 = 0$ so, $\cos x$, $\sin x$ are sol^{n}s of $y'' + y = 0$

To verify superposition principle,

Let, $y = c_1\cos x + c_2\sin x$. ($c_1, c_2 \to$ arb. const.)

$$y'' + y = (-c_1\cos x - c_2\sin x) + (c_1\cos x + c_2\sin x)$$
$$= 0.$$

$\therefore$ $(c_1\cos x + c_2\sin x)$ is a solⁿ of $y'' + y = 0$

<u>Fundamental</u> theorem for <u>the homogeneous linear ODE.</u>

For a <u>homogeneous</u> Linear ODE $y'' + p(x)y' + q(x)y = 0$ — ①

Then any linear combination of two solⁿs of equⁿ ① is again a solⁿ of equⁿ ①

[This can be proved easily by letting y_1, y_2 as two solⁿs of ①]

Note. Fundamental theorem is not applicable for non-homogeneous equⁿ.

Ex: Let, $y_1 = 1 + \cos x$, $y_2 = 1 + \sin x$ are solⁿs of $y'' + y = 1$ (verify). Then check the validation of superposition principle/ fundamental th.

[$y = c_1(1+\cos x) + c_2(1+\sin x)$
$y'' + y = c_1 + c_2$
Take, $c_1 = 1$, $c_2 = 2$
$c_1 + c_2 = 3 \neq 1$]

Ex. Let, $y_1 = x^2$, $y_2 = 1$ are solⁿs of non linear ODE $y''y - xy' = 0$ (check). Then check the validation of superposition principle/ fund..

Note. Fundamental theorem is not applicable for non linear ODE.

<u>Wronskian.</u>

Let, $y_1(x)$ and $y_2(x)$ are two solutions of 2nd order linear DE, then Wronskian of y_1 and y_2 is $W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$

Example Let, $y_1 = e^x$ and $y_2 = e^{2x}$ are two sol^{n}s

then, $W(y_1, y_2) = \begin{vmatrix} e^x & e^{2x} \\ e^x & 2e^{2x} \end{vmatrix} = e^{3x}$

Result. ① Two sol^{n}s $y_1(x)$ and $y_2(x)$ of the eq^n $a_0 y'' + a_1 y' + a_2 y = 0$, $a_0(x) \neq 0$ are linearly indep. iff $W(y_1, y_2) \neq 0$ on (a,b).

② Two " " " dep. iff $W(y_1, y_2)$ is identically zero.

Th. The n- sol^{n}s $y_1, y_2, \ldots, y_n$ of $\frac{d^n y}{dx^n} + P_1 \frac{d^{n-1}y}{dx^{n-1}} + \cdots + P_{n-1}\frac{dy}{dx} + P_n y = 0$ are l.i. on $[a,b]$ if $W(y_1, y_2, \ldots, y_n) \neq 0$ $\forall x \in [a,b]$

where, $W(y_1, y_2, \ldots, y_n) = \begin{vmatrix} y_1 & y_2 & \cdots & y_n \\ y_1' & y_2' & \cdots & y_n' \\ - & - & & - \\ y_1^{n-1} & y_2^{n-1} & \cdots & y_n^{n-1} \end{vmatrix} \neq 0$

Prob. P.t. the sol^n e^x, e^{-x}, e^{2x} of the eq^n $\frac{d^3y}{dx^3} - 2\frac{d^2y}{dx^2} - \frac{dy}{dx} + 2y = 0$ are l.i. on every real interval.

[sol. $W = \begin{vmatrix} e^x & e^{-x} & e^{2x} \\ e^x & -e^{-x} & 2e^{2x} \\ e & e^{-x} & 4e^{2x} \end{vmatrix} = -6e^{2x} \neq$ for all real x

Prob: P.t. the sol^n e^x, xe^x, x^2e^{x} are l.i. for the eq^n $\frac{d^3y}{dx^3} - 3\frac{d^2y}{dx^2} + 3\frac{dy}{dx} - y = 0$

Linear Diff. eq^n with constant coeff.

We ~~shall~~ consider a linear diff. eq^n with constant coeff. $\frac{d^n y}{dx^n} + P_1 \frac{d^{n-1}y}{dx^{n-1}} + \cdots + P_{n-1}\frac{dy}{dx} + P_n y = f(x)$ → ①

(where $P_1, P_2, \ldots, P_n$ are constants)

To find the general sol^n of ①, we have to find its C.F. (i.e. genl. sol^n of $\frac{d^n y}{dx^n} + P_1\frac{d^{n-1}y}{dx^{n-1}} + \cdots + P_n y = 0$ → ②) and a particular integral (P.I) of ①.

$\boxed{\text{Methods of finding C.F.}}$

Now, ~~from~~ ② can be rewritten as $(D^n + P_1 D^{n-1} + P_2 D^{n-2} + \cdots + P_{n-1} D + P_n) y = 0$ (when, $D^n \equiv \frac{d^n}{dx^n}$)

i.e. $F(D) y = 0$ where $F(D) \equiv D^n + P_1 D^{n-1} + \cdots + P_{n-1} D + P_n$

Let, $y = e^{mx}$ be a solⁿ of ②.

Then, A·E is $m^n + P_1 m^{n-1} + P_2 m^{n-2} + \cdots + P_{n-1} m + P_n = 0$ → ③

Characteristic eqⁿ

Case-I $m_1, m_2, \ldots, m_n$ are distinct real roots. Then g.s. of ② is

$$y = c_1 e^{m_1 x} + c_2 e^{m_2 x} + \cdots + c_n e^{m_n x}$$

Case-II Repeated roots.

Let, $m_1 = m_2$ and all other roots are real and distinct. Then g.s. of ② is

$$y = (c_1 + c_2 x) e^{m_1 x} + c_3 e^{m_3 x} + \cdots + c_n e^{m_n x}$$

Let, $m_1 = m_2 = m_3$.

$$y = (c_1 + c_2 x + c_3 x^2) e^{m_1 x} + c_4 e^{m_4 x} + \cdots + c_n e^{m_n x}$$

($c_1, \ldots, c_n$ are arb. const.)

Case-III Supp. A.E. ③ has a pair of complex roots, (say) $\alpha \pm i\beta$ then the g.s. of ② is

$$e^{\alpha x}(c_1 \cos \beta x + c_2 \sin \beta x)$$

If pair of complex roots $(\alpha \pm i\beta)$ occurs twice, then, the g.s. of ② is

$$e^{\alpha x}\left[(c_1 + c_2 x)\cos \beta + (c_3 + c_4 x)\sin \beta x\right].$$

Rules for finding P.I.

Consider the D.E. $F(D)y = f(x)$.

We use the expression $\frac{1}{F(D)} f(x)$ to denote a function (say) $g(x)$, which doesnot contain any arb. constant which gives $f(x)$ when operated upon by $F(D)$.

i.e. $F(D)\left\{\frac{1}{F(D)} f(x)\right\} = f(x)$

$\therefore F(D)\, g(x) = f(x)$

Thus $g(x)$ is a P.I. of the eqⁿ $F(D)y = f(x)$

i.e. $y_p = \frac{1}{F(D)} f(x)$

Rule

① If $F(D) = D$, then $\frac{1}{F(D)} f(x) = \frac{1}{D} f(x)$.

$$= \int f(x)\,dx.$$

② If $F(D) = D - a$, then $\frac{1}{D-a} f(x) = e^{ax}\int f(x)\cdot e^{-ax}\,dx$.

③ If $F(D) = (D-m_1)(D-m_2)\cdots(D-m_n)$.

then, $$y_p = \frac{1}{(D-m_1)(D-m_2)\cdots(D-m_n)} f(x).$$

$$= \frac{1}{(D-m_1)\cdots(D-m_{n-1})}\left\{\frac{1}{(D-m_n)} f(x)\right\}$$

$$= \frac{1}{(D-m_1)\cdots(D-m_{n-1})}\left\{e^{m_n x}\int f(x)e^{m_n x}\,dx\right\}$$

& so on

Short method for finding P.I.

① Let, $f(x) = e^{ax}$ $(a = \text{const.})$, $y_p = \frac{1}{F(D)}e^{ax} = \frac{e^{ax}}{F(a)}$, $F(a) \neq 0$

② Let, $f(x) = x^m$ $(m \rightarrow +\text{ve integer})$

$$y_p = \frac{1}{F(D)}x^m.$$

Expand $\frac{1}{F(D)}$ and arranged the term in ascending powers of D and operate on x^m.

③ $f(x) = \sin ax / \cos ax$.

$$y_p = \frac{1}{F(D)} \sin ax/\cos ax$$

$$= \frac{1}{\phi(D^2)} \sin ax/\cos ax = \frac{\sin ax/\cos ax}{\phi(-a^2)},$$

where $\phi(-a^2) \neq 0$

④ $f(x) = e^{ax}U$ (U is a fn. of x)

$$y_p = \frac{1}{F(D)}e^{ax}U = e^{ax}\frac{1}{F(D+a)}U$$